Q. How does the arithmetic and calculus of power series (as functions) work?

Thm: (Combining/Modifying Power Series) Suppose $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for all x in $(-R_1, R_1)$ and $g(x) = \sum_{n=0}^{\infty} d_n x^n$ converges for all x in $(-R_2, R_2)$. Let $R = \min(R_1, R_2)$. Then

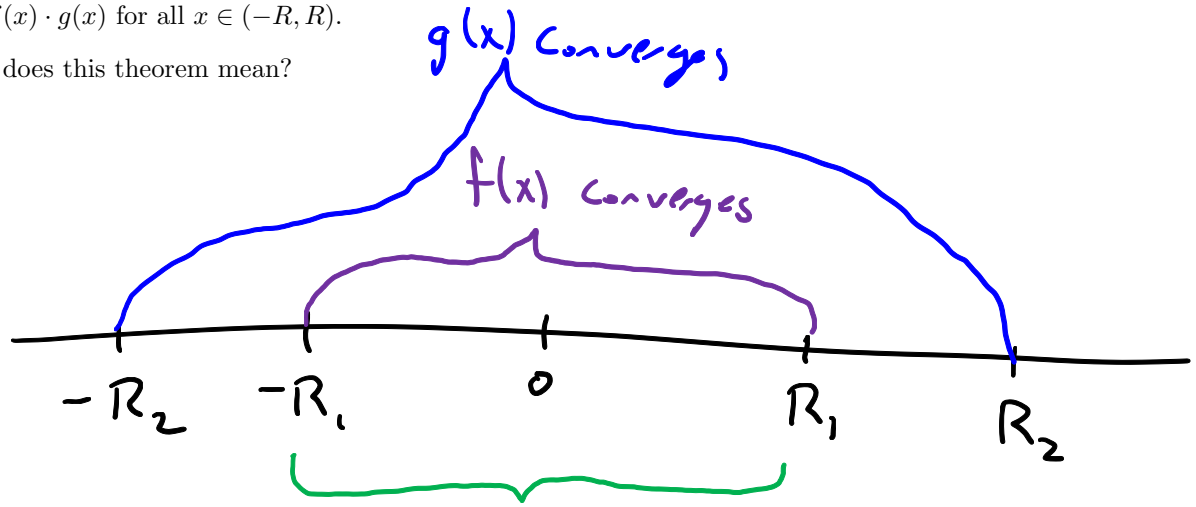
(a) The series $\sum_{n=0}^{\infty} (c_n \pm d_n)x^n$ converges to $f(x) \pm g(x)$ for all $x \in (-R, R)$.

(b) Given a constant b and an integer $m \geqslant 0$, the series $\sum_{n=0}^{\infty} bc_n x^{n+m} = \sum_{n=0}^{\infty} (bx^m)c_n x^n$ converges to $bx^m f(x)$ for all x in $(-R, R)$.

(c) Given a constant b and an integer $m \geqslant 0$, the series $\sum_{n=0}^{\infty} b^n c_n x^{mn} = \sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ for all x such that $bx^m \in (-R, R)$.

(d) If $e_n = \sum_{k=0}^{n} c_k d_{n-k} = c_0 d_n + c_1 d_{n-1} + \cdots + c_n d_0$, then the series $\sum_{n=0}^{\infty} e_n x^n$ converges to the product $f(x) \cdot g(x)$ for all $x \in (-R, R)$.

Q. What does this theorem mean?



(a) $(f \pm g)(x)$ converges

(b) $bx^m f(x)$ converges

(c) $f(bx^m)$ converges

(d) $(f \cdot g)(x)$ converges

Moral: Arithmetic works the way you want it to when all involved series converge

Why is (d) true?

$$f(x)\cdot g(x)=\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right)$$

$$=\left(c_0+c_1x+c_2x^2+c_3x^3+\cdots\right)\left(d_0+d_1x+d_2x^2+\cdots\right)$$

$$= c_0d_0 + c_0d_1x + c_0d_2x^2 + c_0d_3x^3+\cdots$$
$$+ c_1d_0x + c_1d_1x^2 + c_1d_2x^3+\cdots$$
$$+ c_2d_0x^2 + c_2d_1x^3+\cdots$$
$$+ c_3d_0x^3+\cdots$$

$$= c_0d_0 + (c_0d_1+c_1d_0)x + (c_0d_2+c_1d_1+c_2d_0)x^2$$
$$+ (c_0d_3+c_1d_2+c_2d_1+c_3d_0)x^3+\cdots$$
$$+ (c_0d_n+c_1d_{n-1}+c_2d_{n-2}+\cdots+c_nd_0)x^n+\cdots$$

Ex 1 Suppose $f(x) = \dfrac{1}{5-x}$ and $g(x) = \dfrac{1}{x-2}$.

(a) Find the power series for f and g centered at 0.

$$f(x) = \frac{1}{5-x} = \frac{1}{5} \cdot \frac{1}{1-\frac{x}{5}} = \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$$

$$= \frac{1}{5} \sum_{n=0}^{\infty} \frac{x^n}{5^n} = \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}$$

$$g(x) = \frac{1}{x-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{x}{2}}$$

$$= \frac{1}{-2} \cdot \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{-x^n}{2^{n+1}}$$

(b) Use part (a) to find the power series for $\frac{1}{5-3x^7}$ centered at 0.

$$f(x) = \frac{1}{5-x} = \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}$$

$$\frac{1}{5-3x^7} = f(3x^7) = \sum_{n=0}^{\infty} \frac{(3x^7)^n}{5^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{3^n x^{7n}}{5^{n+1}}$$

(c) Use part (a) to find the power series for $\frac{3x}{x^2-2}$ centered at 0.

$$\frac{1}{x-2} = g(x) = \sum_{n=0}^{\infty} -\frac{x^n}{2^{n+1}}$$

$$\frac{3x}{x^2-2} = 3x \cdot \frac{1}{x^2-2} = 3x \cdot g(x^2)$$

$$= 3x \sum_{n=0}^{\infty} -\frac{(x^2)^n}{2^{n+1}}$$

$$= 3x \sum_{n=0}^{\infty} \frac{-x^{2n}}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{-3x^{2n+1}}{2^{n+1}}$$

(d) Write out the first 3 terms of the series for $\frac{1}{(5-x)(x^2-2)}$ centered at 0.

$$\frac{1}{(5-x)(x^2-2)} = \frac{1}{5-x} \cdot \frac{1}{x^2-2}$$

$$= \left(\sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{(x^2)^n}{2^{n+1}}\right)$$

$$= \left(\frac{1}{5} + \frac{x}{25} + \frac{x^2}{125} + \cdots\right)\left(-\frac{1}{2} - \frac{x^2}{4} - \frac{x^4}{8} - \cdots\right)$$

$$= -\frac{1}{10} - \frac{x^2}{20} - \frac{x^4}{40} - \cdots$$

$$- \frac{x}{50} - \frac{x^3}{100} - \frac{x^5}{200} - \cdots$$

$$- \frac{x^2}{250} - \frac{x^4}{500} - \cdots$$

$$= -\frac{1}{10} - \frac{x}{50} - \left(\frac{1}{20} + \frac{1}{250}\right)x^2 - \cdots$$

Ex 2 Find the series $\frac{1}{(5-x)(x-2)}$ centered at 0 using partial fraction decomposition.

$$\frac{1}{(5-x)(x-2)} = \frac{A}{5-x} + \frac{B}{x-2}$$

$$1 = A(x-2) + B(5-x)$$

$x=2$: $1 = B\cdot 3 \to B = \frac{1}{3}$

$x=5$: $1 = A\cdot 3 \to A = \frac{1}{3}$

$$\frac{1}{(5-x)(x-2)} = \frac{1/3}{5-x} + \frac{1/3}{x-2}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} + \frac{1}{3}\sum_{n=0}^{\infty}\frac{-x^n}{2^{n+1}}$$

Converges on $(-5, 5)$

Converges on $(-2, 2)$

$$= \sum_{n=0}^{\infty}\frac{x^n}{3\cdot 5^{n+1}} + \sum_{n=0}^{\infty}\frac{-x^n}{3\cdot 2^{n+1}}$$

$$= \left(\frac{1}{3\cdot 5} + \frac{x}{3\cdot 5^2} + \frac{x^2}{3\cdot 5^3} + \cdots\right) + \left(\frac{-1}{3\cdot 2} - \frac{x}{3\cdot 2^2} - \frac{x^2}{3\cdot 2^3} - \cdots\right)$$

$$= \left(\frac{1}{3\cdot 5} - \frac{1}{3\cdot 2}\right) + \left(\frac{1}{3\cdot 5^2} - \frac{1}{3\cdot 2^2}\right)x + \left(\frac{1}{3\cdot 5^3} - \frac{1}{3\cdot 2^3}\right)x^2 + \cdots$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{3\cdot 5^{n+1}} - \frac{1}{3\cdot 2^{n+1}}\right)x^n$$

→ Converges on $(-2, 2)$

Q. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence R. What is the radius of convergence of $\sum_{n=1}^{\infty} n \cdot c_n x^{n-1}$? What about $\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$? Can you tell anything about the intervals of convergence for these two series?

Radius of convergence ↓ R.O.C. $= R$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$\sum_{n=1}^{\infty} n \cdot c_n \cdot x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1} = c_0 x + \frac{c_1}{2} x^2 + \frac{c_2}{3} x^3 + \cdots$$

$$\lim_{n\to\infty} \left| \frac{c_{n+1} x^{n+1}}{c_n x^n} \right| = |x| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| < 1$$

$$|x| < \frac{1}{\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|} \longleftrightarrow x \in \left(\frac{-1}{\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|}, \frac{1}{\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|} \right)$$

$$\to R = \frac{1}{\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|} \to \frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

Q: Will the R.o.C of $\sum_{n=1}^{\infty} n \cdot c_n \cdot x^{n-1}$ be R?

a) bigger
b) smaller
c) the same as

$$\lim_{n\to\infty} \left| \frac{(n+1) \cdot c_{n+1} \cdot x^n}{n \cdot c_n \cdot x^{n-1}} \right| = |x| \cdot \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{c_{n+1}}{c_n} \right| = |x| \frac{1}{R} < 1$$

($\frac{n+1}{n} \to 1$, $\frac{c_{n+1}}{c_n} \to \frac{1}{R}$)

$$\longleftrightarrow |x| < R \longleftrightarrow x \in (-R, R)$$

↳ R.o.C. of $\sum_{n=0}^{\infty} c_n x^n$ agrees with R.o.C. of $\sum_{n=0}^{\infty} n \cdot c_n x^{n-1}$, but the I.o.C. might not agree

You check: R.o.C. of $\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$ agrees with R.o.C. of $\sum_{n=0}^{\infty} c_n x^n$

Thm: (Term-by-term Differentiation and Integration) Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ converges for all $x \in (a-R, a+R)$. Then for all $x \in (a-R, a+R)$, f is differentiable and integrable and

$$\frac{df}{dx} = \sum_{n=1}^{\infty} n \cdot c_n \cdot (x-a)^{n-1}$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

Ex 3 Find the series for $f(x) = \frac{1}{(1-x)^2}$ centered at 0.

$$f(x) = \left(\frac{1}{1-x}\right)^2 = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right) = \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} (n+1)x^n$$

$$= 1 + 2x + 3x^2 + \cdots$$

$\forall n: c_n = 1 \qquad d_n = 1$

$$e_n = c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_n d_0 = 1 + 1 + 1 + \cdots + 1 = n+1$$

($n+1$ terms)

Note: $\frac{1}{(1-x)^2} = \frac{d}{dx}\left((1-x)^{-1}\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} n x^{n-1}$

$$= 1 + 2x + 3x^2 + \cdots$$

Ex 4 Find the series for $f(x) = \ln|1-x|$ centered at 0.

$$\frac{df}{dx} = \frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n$$ ← converges for $x \in (-1, 1)$

$$\downarrow$$

$$f(x) = \int -\sum_{n=0}^{\infty} x^n \, dx = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

To find C: check an easy x-value

$$f(0) = C - 0 = C$$

$$0 = \ln|1-0|$$

$$f(x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$\ln(a^b) = b\ln(a)$

$\frac{1}{n^2} \to 0$

$\frac{\ln(n)}{n} \to 0$

Ex 5 Use power series to estimate ln(2).

Q. Why not $\ln|x|$ centered at 0?

↳ $\ln 0$ doesn't exist

$$\ln(2) = f(-1) \neq -\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

$$= -\left(-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \cdots\right)$$

$$\ln(2) = -\ln\left(\frac{1}{2}\right) = -f\left(\frac{1}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(1/2)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)}$$

$f(x) = x^3(2-x)$

$f'(x) = 3x^2(2-x) + x^3(-1)$

$f''(x) = 6x(2-x) + 3x^2(-1) - 3x^2$

$f'''(x) = 6(2-x) + 6x(-1) - 6x - 6x$

$g(x) = x(2-x)$

$g'(x) = (2-x) + x(-1)$